

Measurements in the Lévy quantum walk

A. Romanelli*

*Instituto de Física, Facultad de Ingeniería
Universidad de la República
casilla de correo 30, código postal 11000,
Montevideo, Uruguay*
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We study the quantum walk subjected to measurements with a Lévy waiting-time distribution. We find that the system has a sub-ballistic behavior instead of a diffusive one. We obtain an analytical expression for the exponent of the power law of the variance as a function of the characteristic parameter of the Lévy distribution.

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I. INTRODUCTION

The development of the quantum walk (QW) in the context of quantum computation, as a generalization of the classical random walk, has attracted the attention of researchers from different fields. The fact that it is possible to build and preserve quantum states experimentally has led the scientific community to think that quantum computers could be a reality in the near future. On the other hand, from a purely physical point of view, the study of quantum computation allows to analyze and verify the principles of quantum theory. In this last frame the study of the QW subjected to different sources of decoherence is a topic that has been considered by several authors [1]. In particular we have recently studied [2] the QW and the quantum kicked rotor in resonance subjected to noise with a Lévy waiting-time distribution [3], finding that both systems have a sub-ballistic wave function spreading, as shown by the power-law tail of the standard deviation ($\sigma(t) \sim t^c$ with $0.5 < c < 1$), instead of the known ballistical growth ($\sigma(t) \sim t$). This sub-ballistic behavior was also observed in the dynamics of both the quantum kicked rotor [4] and the QW [5] when these systems are subjected to an excitation that follows an aperiodic Fibonacci prescription. Other authors also investigated the kicked rotor subjected to noises with a Lévy distribution [6] and almost-periodic Fibonacci sequence [7], showing that this decoherence never fully destroys the dynamical localization of the kicked rotor but leads to a sub-diffusion regime for a short time before localization appears. All these mentioned papers have in common that they work with quantum systems that have an anomalous behavior that was established numerically. There are no analytical results that explain in a general way why noises, with a power-law distribution or in a Fibonacci sequence, lead the system to a new non-diffusive behavior. Here we present a simple model that allows an analytical treatment to understand the sub-ballistic behavior. We hope that this may help to understand in

a generical way how the frequency of the decoherence is the main factor in this unexpected dynamics. With this aim we investigate the QW when measurements are performed on the system with waiting times between them following a Lévy power-law distribution. We show that this noise produces a change from ballistic to sub-ballistic behavior and we obtain analytically a relation between the exponent of the standard deviation and the characteristic parameter of Lévy distribution.

The paper is organized as follows. In the next section we develop the QW model with Lévy noise, in the third section analytical results are obtained and in the last section we draw the conclusions.

II. QUANTUM WALK AND MEASUREMENT

The dynamics of the QW subjected to a series of measurements will be generated by a large sequence of two time-step unitary operators U_0 and U_1 as was done in a previous work [2]. But now U_0 is the ‘free’ evolution of the QW and U_1 is the operator that measures simultaneously the position and the chirality of the QW. The time interval between two applications of the operator U_1 is generated by a waiting-time distribution $\rho(T)$, where T is a dimensionless integer time step. The detailed mechanism to obtain the evolution is given in [2]. We take $\rho(T)$ in accordance with the Lévy distribution [8, 9, 10] that includes a parameter α , with $0 < \alpha \leq 2$. When $\alpha < 2$ the second moment of ρ is infinite, when $\alpha = 2$ the Fourier transform of ρ is the Gaussian distribution and the second moment is finite. Then, this distribution has no characteristic size for the temporal jump, except in the Gaussian case. The absence of scale makes the Lévy random walks scale-invariant fractals. This means that any classical trajectory has many scales but none in particular dominates the process. The most important characteristic of the Lévy noise is the power-law shape of the tail, accordingly in this work we use the waiting-time distribution

$$\rho(t) = \frac{\alpha}{(1+\alpha)} \begin{cases} 1, & 0 \leq t < 1 \\ (\frac{1}{t})^{\alpha+1}, & t \geq 1 \end{cases}. \quad (1)$$

*E-mail address: alejo@fing.edu.uy

To obtain the time interval T we sort a continuous variable t in agreement with eq. (1) and then we take the integer part T_i of this variable [2].

To obtain the operator U_0 we develop in some detail the free QW model. The standard QW corresponds to a one-dimensional evolution of a quantum system (the walker) in a direction which depends on an additional degree of freedom, the chirality, with two possible states: ‘left’ $|L\rangle$ or ‘right’ $|R\rangle$. Let us consider that the walker can move freely over a series of interconnected sites labeled by an index n . In the classical random walk, a coin flip randomly selects the direction of the motion; in the QW the direction of the motion is selected by the chirality. At each time step a rotation (or, more generally, a unitary transformation) of the chirality takes place and the walker moves according to its final chirality state. The global Hilbert space of the system is the tensor product $H_s \otimes H_c$ where H_s is the Hilbert space associated to the motion on the line and H_c is the chirality Hilbert space.

If one is only interested in the properties of the probability distribution it suffices to consider unitary transformations which can be expressed in terms of a single real angular parameter θ [11, 12, 13]. Let us call M_- (M_+) the operators that move the walker one site to the left (right) on the line in H_s and let $|L\rangle\langle L|$ and $|R\rangle\langle R|$ be the chirality projector operators in H_c . Then we consider free evolution transformations of the form [11],

$$U_0(\theta) = \{M_- \otimes |L\rangle\langle L| + M_+ \otimes |R\rangle\langle R|\} \circ \{I \otimes K(\theta)\}, \quad (2)$$

where $K(\theta) = \sigma_z e^{-i\theta\sigma_y}$ is an unitary operator acting on H_c , σ_y and σ_z being the standard Pauli matrices, and I is the identity operator in H_s . The unitary operator $U_0(\theta)$ evolves the state $|\Psi(t)\rangle$ by one time step,

$$|\Psi(t+1)\rangle = U_0(\theta)|\Psi(t)\rangle. \quad (3)$$

The wave-vector $|\Psi(t)\rangle$ is expressed as the spinor

$$|\Psi(t)\rangle = \sum_{n=-\infty}^{\infty} \begin{pmatrix} a_n(t) \\ b_n(t) \end{pmatrix} |n\rangle, \quad (4)$$

where we have associated the upper (lower) component to the left (right) chirality, the states $|n\rangle$ are eigenstates of the position operator corresponding to the site n on the line. The unitary evolution for $|\Psi(t)\rangle$, corresponding to eq. (3) can then be written as the map

$$\begin{aligned} a_n(t+1) &= a_{n+1}(t) \cos \theta + b_{n+1}(t) \sin \theta, \\ b_n(t+1) &= a_{n-1}(t) \sin \theta - b_{n-1}(t) \cos \theta. \end{aligned} \quad (5)$$

To build the measurement operator U_1 we consider the case in which the walker starts from the position eigenstate $|0\rangle$, and with an initial qubit state $(a_0, b_0) = (1, i)/\sqrt{2}$. The operator U_1 must describe the measurement of position and chirality simultaneously. The measurement of position is direct, but among the many ways to measure the chirality we choose to do it in such a way

that the two qubit states $(1, i)/\sqrt{2}$, and $(1, -i)/\sqrt{2}$ are eigenstates of the measurement operator. This means that we project the chirality on the y direction using the σ_y Pauli operator. This form of measurement ensures that the initial conditions after each measurement are equivalent to each other from the point of view of the probability distribution [14]. In this work we take $\theta = \pi/4$ as in the usual Hadamard walk on the line. The probability distribution for the walker’s position at time t is given by

$$P_n(t) = |a_n(t)|^2 + |b_n(t)|^2. \quad (6)$$

The effect of performing measurements on this system at time intervals T_i , with a Lévy distribution, combined with unitary evolution is as follows: the standard deviation has a ballistic growth during the free evolution, and when a measurement collapses the wave function the standard deviation starts again from zero. Then the standard deviation has a zig-zag path. In Fig. 1 we present

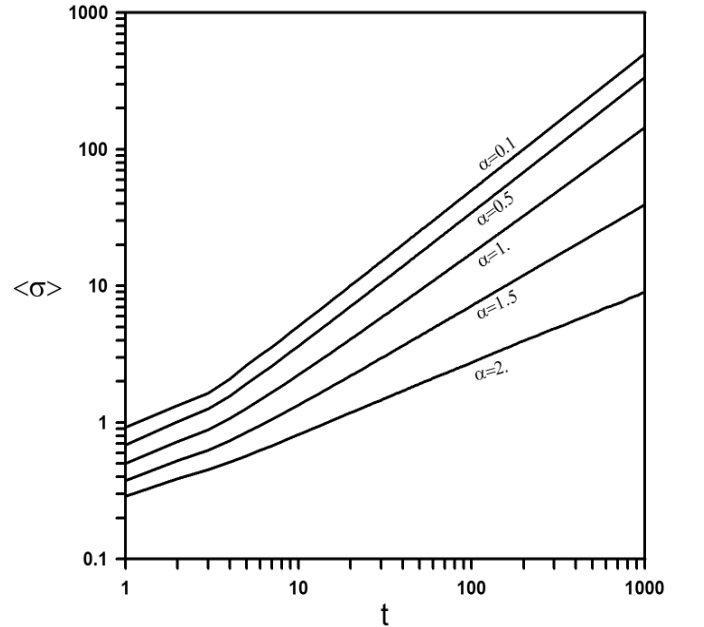


FIG. 1: The standard deviation for the QW as a function of dimensionless time in logarithmic scales. The parameters of the curves from top to bottom are: (1) $\alpha = 0.1$ and $c = 1.$; (2) $\alpha = 0.5$ and $c = 0.99$; (3) $\alpha = 1.$ and $c = 0.92$; (4) $\alpha = 1.5$ and $c = 0.74$ and (5) $\alpha = 2.$ and $c = 0.51$.

the numerical calculation of the average standard deviation $\langle \sigma(t) \rangle$ of the QW with measurements, calculated through a computer simulation of the time evolution of an ensemble of 2×10^6 stochastic trajectories for each value of the parameter α . This figure shows for different values of α that the behavior is diffusive ($\sim t$) for times $t \lesssim 10$ and follows a power law $\sim t^c$ for times $t \gtrsim 10$. One would expect that the large degree of decoherence introduced by measurements led to a diffusive behavior for all times [2]; instead the system changes to an unexpected

sub-ballistic regime. Then α determines the degree of diffusivity of the system for large waiting times, that is ballistic for $\alpha = 0$, sub-ballistic for $0 < \alpha < 2$ and diffusive for $\alpha = 2$. Fig. 2 shows, in full line, the exponent c of the power law as a function of the Lévy parameter α . Again the calculation has been made with an ensemble of 2×10^6 stochastic trajectories for each value of α . The passage from the ballistic behavior for $\alpha = 0$ to the diffusive behavior for $\alpha = 2$ is clearly observed. In this figure the result of the analytical calculation of the next section is also presented in dashed line, the comparison of the curves proving the coherence of both treatments.

III. THEORETICAL MODEL

Different mechanisms of unitary noise may drive a system from a quantum behavior at short times to a classical-like one, at longer times. It is clear that the quadratic growth in time of the variance of the QW is a direct consequence of the coherence of the quantum evolution [15]. In this section, we develop an analytical treatment to understand why the Lévy decoherence does not fully break the ballistic behavior.

In our previous work [16] we investigated the QW on the line when decoherence was introduced through simultaneous measurements of the chirality and position. In that work it was proved that the QW shows a diffusive behavior when measurements are made at periodic times or with a Gaussian distribution. Now, we use the Lévy distribution but some of the result obtained in [16] can be used. For the sake of clarity we reproduce briefly the main steps to obtain the dynamical equation of the variance. Let us suppose that the wave-function is measured at the time t , then it evolves according to the unitary map (5) during a time interval T , and again at this last time $t + T$, a new measurement is performed. The probability that the wave-function collapses in the eigenstate $|n\rangle$ due to the position measured, after a time T , is

$$q_n \equiv P_n(T). \quad (7)$$

These spatial distributions q_n depend on the initial qubit state and the time interval T , they will play the role of transition probabilities for the global evolution. The mechanism used to perform measurements of position and chirality assures that these distributions will repeat themselves around the new position, because the initial chirality $((1, i)/\sqrt{2}$ or $(1, -i)/\sqrt{2})$ produces the same spatial distributions q_n and their value only depends on the size of T . Then it is straightforward to build the probability distribution P_n at the new time $t + T$ as a convolution between this distribution at the time t with the conditional probability $q_n(T)$; this takes the form of the following ‘master equation’

$$P_n(t + T) = \sum_{j=n-T}^{n+T} q_{n-j} P_j(t), \quad (8)$$

where q_{n-j} are the transition probabilities from site j to site n , defined in eq. (7) and the sum is extended between $j = n - T$ and $j = n + T$, because T is also used as the number of applications of the quantum map eq. (5). Remember that for each time step the walker moves one spatial step to the right and left. Using eq. (8) we calculate the first moment $m_1(t) \equiv \sum j P_j(t)$ and the second moment $m_2(t) \equiv \sum j^2 P_j(t)$ to obtain

$$m_1(t + T) = m_1(t) + m_{1q}(T) \quad (9)$$

$$m_2(t + T) = m_2(t) + 2m_1(t)m_{1q}(T) + m_{2q}(T) \quad (10)$$

where $m_{1q}(T) = \sum_{n=-T}^{n=T} n q_n$ and $m_{2q}(T) = \sum_{n=-T}^{n=T} n^2 q_n$ are the first and second moments of the unitary evolution between measurements. Therefore the global variance

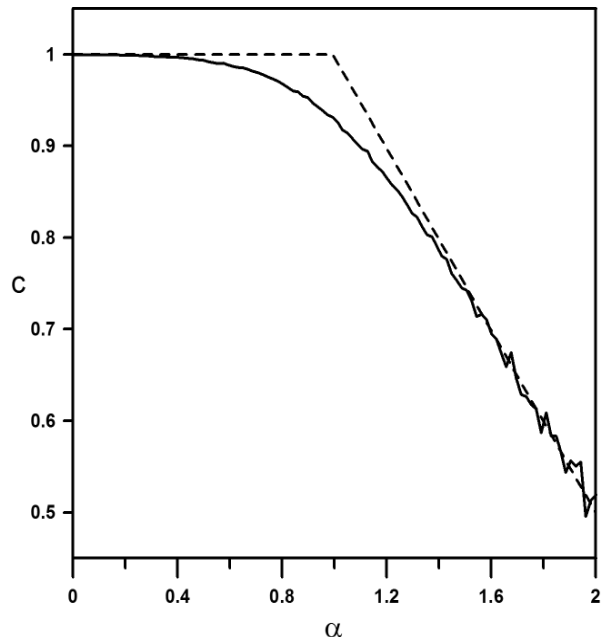


FIG. 2: the exponent c of the power law of the standard deviation for the QW as a function of the parameter α . The full line correspond to the numerical result and the dashed line to the analytical one

$\sigma^2(t) = m_2(t) - m_1^2(t)$ verifies the following equation for the process, obtained for first time in [16]

$$\sigma^2(t + T) = \sigma^2(t) + \sigma_q^2(T), \quad (11)$$

where $\sigma_q^2(T) = m_{2q}(T) - m_{1q}^2(T)$ is the variance associated to the unitary evolution between measurements. These results can be used for random time intervals T between consecutive measurements. The variance σ_q^2 depends very weakly on the qubit's initial conditions and it increases quadratically with time [11]

$$\sigma_q^2(T) = kT^2, \quad (12)$$

where $T \gg 1$ and k is a constant determined by the initial conditions. We now calculate the average of eq.

(11) taking integer time intervals T_i between consecutive measurements sorted according to the Lévy distribution eq. (1)

$$\langle \sigma^2(t + T_i) \rangle = \langle \sigma^2(t) \rangle + k \langle T_i^2 \rangle, \quad (13)$$

where $\langle f(t) \rangle \equiv \int_0^t f(x) \rho(x) dx$. From the previous numerical calculation, we know that the variance grows as a power law, then $\langle \sigma^2(t) \rangle \propto t^{2c}$ and $\langle \sigma^2(t + T_i) \rangle \propto \langle (t + T_i)^{2c} \rangle \approx t^{2c}(1 + 2c \frac{\langle T_i \rangle}{t})$ for large t . Substituting these expressions in eq. (13), the following result for the exponent c is obtained

$$c \approx \frac{1}{2} \left(1 + \frac{\log \frac{\langle T_i^2 \rangle}{\langle T_i \rangle}}{\log t} \right). \quad (14)$$

valid for a large t . The first and the second moments of the waiting time of our Lévy distribution are

$$\langle T_i \rangle = \frac{\alpha}{\alpha + 1} \left\{ 1 + \frac{t^{1-\alpha} - 1}{1 - \alpha} \right\}, \quad (15)$$

$$\langle T_i^2 \rangle = \frac{\alpha}{\alpha + 1} \left\{ \frac{1}{3} + \frac{t^{2-\alpha} - 1}{2 - \alpha} \right\}. \quad (16)$$

Therefore, in the case when $t \rightarrow \infty$ the exponent c is

$$c = \begin{cases} 1, & \text{if } 0 \leq \alpha \leq 1 \\ \frac{1}{2}(3 - \alpha), & \text{if } 1 \leq \alpha \leq 2. \end{cases} \quad (17)$$

This result is in accordance with the numerical result obtained in the previous section, as can be seen in Fig. 2. It is important to remark that eq. (17) gives the analytical dependence of the exponent of the power law for $\langle \sigma^2 \rangle$ on the parameter α .

IV. CONCLUSION

Several systems have been proposed as candidates to implement the QW model, they include atoms trapped in optical lattices [17, 18], cavity quantum electrodynamics [19] and nuclear magnetic resonance in solid substrates [20, 21]. All these proposed implementations face the obstacle of decoherence due to environmental noise and imperfections. Thus the study of the QW subjected to different types of noise may be important in future technical applications. Here we showed that the QW subjected to measurements with a Lévy waiting-time distribution does not break completely the coherence in the dynamics, but produces a sub-ballistic behavior in the system, as an intermediate situation between the ballistic and the diffusive behavior. Note that as Gaussian noise is a particular case of the Lévy noise, our study is open to wider experimental situations. We studied this behavior numerically and we obtained also an analytical expression for the exponent of the power law of the variance as a function of the characteristic parameter of the Lévy distribution. Measurement is one of the strongest possible decoherences on any quantum system, but even in this case the coherence of the QW treated here is not fully lost. This fact shows that the most important ingredient for the sub-ballistic behavior is the temporal sequence of the perturbation, not its size or type. This extreme and simple model that can be treated analytically allows us to understand why similar models [2, 4, 5] (that were studied numerically) show also a power-law behavior.

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